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Prolongation structures of complex quasi-polynomial evolution equations

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Abstract. We use Wahlquist–Estabrook prolongation theory to investigate second-order complex equations of generalised NLS–DNLS type. We isolate a number of new integrable cases.

1. Introduction

In 1975 Wahlquist and Estabrook (Wahlquist and Estabrook 1975) introduced their prolongation method of finding the linear scattering problem (assuming one exists) associated with a given nonlinear evolution equation. In 1976 (Wahlquist and Estabrook 1976) they applied their method to the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t = u_{xx} + 2u^2\bar{u} \tag{1.1}$$

where $\bar{u}$ is the complex conjugate of u and $\mathrm{i} = \sqrt{-1}$. They gave a 'systematic' derivation of the result of Zakharov and Shabat (1972), who had introduced the linear scattering problem

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}\lambda & u\\ -\bar{u} & -\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{1.2}$$

corresponding to (1.1). Since 1976 there has grown quite a long list of papers which either elaborate on the geometrical structure of prolongation theory or apply the method to various partial differential equations (see Kaup 1980 and references therein).

In a recent paper (Dodd and Fordy 1983) which we shall refer to as I, the present authors gave a detailed discussion of the prolongation method, but viewed as a (potential) algorithm. There are two main steps in any prolongation calculation: the first is to start with a differential equation and, after some work, derive a set of generators and relations for an incomplete Lie algebra (the prolongation algebra); the second step is to complete this Lie algebra and find a finite matrix representation for the derived set of generators.

In I we mainly discussed the prolongation structure of real, quasi-polynomial evolution equations. Such equations have the form

$$u_t^\alpha = K^\alpha[u^\beta] \tag{1.3}$$

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where $K^\alpha[u^\beta]$ is a quasi-polynomial function of u^β and their x-derivatives; that is, K^α is a polynomial in the x-derivatives of u^β with coefficients which are almost everywhere C^∞ functions of the u^β themselves. These equations are particularly simple, so we were able to present a genuine algorithm for the first half of the corresponding prolongation calculation. We derived the prolongation structure of several examples of quasi-polynomial flow. Among these was the complex equation

$$\mathrm{i}q_t = q_{xx} + 2\mathrm{i}q\bar{q}q_x \tag{1.4}$$

which is a version of derivative nonlinear Schrödinger equation introduced by Chen *et al* (1979). This equation was presented as an interesting example even though it did not strictly fit into the scheme of our algorithm.

The purpose of the present paper is to investigate the class of second-order equations:

$$u_t = a(u, \bar{u})u_{xx} + b(u, \bar{u})\bar{u}_{xx} + K(u, \bar{u}, u_x, \bar{u}_x) \tag{1.5}$$

and derive a restricted form of K in order that (1.5) be integrable. In addition we extend the algorithm to the case of *complex*, quasi-polynomial flows. This is done in § 2 for the general case although we make no further use of this algorithm in the remainder of this paper and is included for completeness.

Equation (1.5) represents a very broad class of equations and it is beyond the scope and aims of this paper to derive the complete set. The conditions imposed upon K are insufficient to completely determine this function. However, if one uses prolongation theory in conjunction with symmetry arguments it is possible to derive a restricted class of K for which (1.5) is integrable. In particular, in § 3 we use a scaling argument to derive a wide class of such equations. This includes all the previously known single component complex equations, but several others. In particular, we derive the linear problems for

$$u_t = \mathrm{i}u_{xx} + 2c_2(u^{3/2}\bar{u}^{1/2})_x + c_3u^2\bar{u} \tag{1.6}$$

(equation (3.33)), which is a generalisation of the NLS equation, and (equation (3.29))

$$u_t = \mathrm{i}u_{xx} + c_1u\bar{u}u_x + c_2u^2\bar{u}_x - \tfrac{1}{4}\mathrm{i}c_2(2c_2 - c_1)u^3\bar{u}^2. \tag{1.7}$$

This equation is a generalisation of both the Kaup–Newell DNLS ($c_1 = 2c_2$) and that of Chen *et al* (1979) ($c_2 = 0$). We also show that (equation (3.7))

$$u_t = \mathrm{i}u_{xx} - (2\mathrm{i}/\bar{u})u_x\bar{u}_x \tag{1.8}$$

discussed by Nijhoff *et al* (1982), is simply related to the NLS equation by $Q = u_x/\bar{u}$.

Generalised symmetries are discussed in appendix 1. We derive conditions on K such that (1.5) (with $b \equiv 0$, constant a) possesses a commuting third-order flow. However, this method is not very efficient and to derive the full set of conditions would involve an enormous amount of work.

In our previous paper we discussed prolongation algebras in great detail. For large systems of equations it is necessary to consider arbitrarily large Lie algebras. It was therefore necessary to develop a lot of abstract algebraic machinery to handle large prolongation algebras. The specific equations we deal with in the present paper are all associated with the smallest of simple Lie algebras, sl(2, C) apart from the example of appendix 4. The standard basis for sl(2, C) satisfies the following set of commutation relations:

$$[h, e_+] = 2e_+, \qquad [h, e_-] = -2e_-, \qquad [e_+, e_-] = h. \tag{1.9}$$

In the fundamental representation these elements are given by:

$$e_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad e_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\qquad h=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.10}$$

For this paper the following automorphisms of sl(2, C) will prove useful:

$$e_+\to\lambda e_+,\qquad h\to h,\qquad e_-\to\lambda^{-1}e_- \tag{1.11a}$$

$$e_+\to \mathrm{e}^{\mathrm{i}\theta}e_+,\qquad h\to h,\qquad e_-\to \mathrm{e}^{-\mathrm{i}\theta}e_- \tag{1.11b}$$

where λ and θ are real. These are respectively a real scaling symmetry and a phase symmetry which can be combined to form a 'complex scaling symmetry' see (3.2)).

2. Prolongation structure

We now investigate the existence of a non-trivial prolongation structure for complex quasi-polynomial evolution equations.

Definition. $K[u, \bar{u}]$ is complex quasi-polynomial if it is polynomial in the x-derivatives of u and $\bar{u}$, but not necessarily in the u and $\bar{u}$ themselves.

The usual polynomial K are in a subclass of these.

We consider equations of the form:

$$u_t=a(u,\bar{u})u_{n+1}+b(u,\bar{u})\bar{u}_{n+1}+K(u,\bar{u},u_1,\bar{u}_1,\ldots,u_n,\bar{u}_n) \tag{2.1}$$

together with their complex conjugate. Here, a and b are almost everywhere C^∞ functions of u and $\bar{u}$,

$$u_i\equiv\partial^i u/\partial x^i,\qquad u_0\equiv u \text{ and } K \text{ is quasi-polynomial.}$$

We wish to determine those equations (2.1) which can be represented as the integrability conditions of a pair of linear equations:

$$\phi_x=F\phi,\qquad \phi_t=G\phi \tag{2.2}$$

where F and G are matrices whose size (and Lie algebra) is yet to be determined. The integrability conditions of (2.2) are

$$F_t-G_x+[F,G]=0. \tag{2.3}$$

As explained in (Dodd and Fordy 1983), for evolution equations of the form (2.1) F and G usually have the following functional dependence:

$$F(u,\bar{u}),\qquad G(u,\bar{u},u_1,\bar{u}_1,\ldots,u_n,\bar{u}_n) \tag{2.4}$$

although, as a strict rule this is contradicted by the example (3.7) below. With (2.4) the integrability conditions (2.3) then decouple to give the system of partial differential equations:

$$\begin{aligned}&G_{,u_n}=aF_{,u}+\bar{b}F_{,\bar{u}},\qquad G_{,\bar{u}_n}=\bar{a}F_{,\bar{u}}+bF_{,u}\\ &KF_{,u}+\bar{K}F_{,\bar{u}}+\sum_{i=1}^{n}u_iG_{,u_{i-1}}-\sum_{i=1}^{n}\bar{u}_iG_{,\bar{u}_{i-1}}+[F,G]=0.\end{aligned} \tag{2.5}$$

For a given function K this is an overdetermined system of equations for F and G. These equations can be solved recursively for G in terms of the derivatives u_i and $\bar{u}_i$ During the calculation some differential conditions on F arise. If these are consistent, then we can solve F in terms of u and $\bar{u}$; if not, the equation (2.1) is not completely integrable.

For a given equation this is a straightforward calculation. However, it is not *a priori* evident that the calculation can always be carried through from beginning to end. We next give an algorithm which proves that this can *always* be explicitly carried out. Furthermore, the algorithm is presented in such a way that it could easily be incorporated in an algebraic manipulative computer program.

2.1. The algorithm

For our calculations it is convenient to introduce the algebra $P[u, \bar{u}]$ of polynomials in the indeterminates u_j, $\bar{u}_j$, $j = 1, \ldots, n$ over the ring of almost everywhere C^∞ functions of u, $\bar{u}$ to which the quasi-polynomials belong. We shall also need the algebra $P[u, \bar{u}, F]$ in the indeterminates u_j, $\bar{u}_j$, $F_{i\bar{j}}$, where $F_{i\bar{j}} = \partial^{i+j}F/\partial u^i\, \partial \bar{u}^j$ $i, j = 1, \ldots, n$, over the ring of almost everywhere C^∞ functions of u, $\bar{u}$. The indeterminates $F_{i\bar{j}}$ are elements of a Lie algebra so that we shall denote products in $P[u, \bar{u}, F]$ by the Lie bracket [,]. Elements of $P[u, \bar{u}]$, $P[u, \bar{u}, F]$ are graded by the highest-order u-indeterminate, $u_j > \bar{u}_i$ for any i, j and $u_j > u_i$, $\bar{u}_j > \bar{u}_i$ if $j > i$. It follows that for terms quadratic in the u, $\bar{u}$ indeterminates $u_i u_j > u_l \bar{u}_k$ if $i > l$ or if $i = l$ and $j > k$. This is easily extended to arbitrary monomials in the u, $\bar{u}$ indeterminates, and thence to the whole algebra. For elements of $P[u, \bar{u}]$ which we shall denote by lower case italic letters, superscripts will indicate the highest u and $\bar{u}$ indeterminates in the polynomial. Thus $p^{i\bar{j}}$ means that u_i and $\bar{u}_j$ are the highest order u and $\bar{u}$ indeterminates in $p^{i\bar{j}}$; if $i = j$ then we write p^i for $p^{i\bar{j}}$. Elements of $P[u, \bar{u}, F]$ will be denoted by capital letters. For elements of this algebra it is convenient to also indicate the highest order $F_{i\bar{j}}$ indeterminate which occurs. In this case we make no distinction between the barred and unbarred indices. Thus $H^{(i\bar{j}v)}$ denotes a polynomial in which u_i, $\bar{u}_j$ are the highest u, $\bar{u}$ indeterminates and in which members of the set $\{F_{i\bar{j}}\colon i + j = v\}$ (some possibly absent) are the highest order $F_{i\bar{j}}$ indeterminates ($F_{i\bar{j}} > F_{l\bar{k}}$ if $i + j > l + k$). We shall write $H^{(i,v)}$ for $H^{(i\bar{i}v)}$ and omit the v index if this is unknown. The algebra $P[u, \bar{u}, F]$ becomes a differential algebra under the action of the operators $\partial/\partial u$, ∂/u_j, $\partial/\partial \bar{u}_j$ which are defined by

$$
\frac{\partial}{\partial u}\colon \begin{cases} F_{i\bar{j},u} = F_{i+1\bar{j}} \\ u_{j,u} = 0 \\ \bar{u}_{j,u} = 0 \end{cases}
\qquad
\frac{\partial}{\partial \bar{u}}\colon \begin{cases} F_{i\bar{j},\bar{u}} = F_{i\overline{j+1}} \\ u_{j,\bar{u}} = 0 \\ \bar{u}_{j,\bar{u}} = 0 \end{cases}
$$
$$
\frac{\partial}{\partial u_k}\colon \begin{cases} (u_i)^j{}_{,u_k} = j\delta_{ik}(u_i)^{j-1} \\ \bar{u}_{i,u_k} = 0 \\ F_{i\bar{j},u_k} = 0 \end{cases}
\qquad
\frac{\partial}{\partial \bar{u}_k}\colon \begin{cases} (\bar{u}_i)^j{}_{,\bar{u}_k} = j\delta_{ik}(\bar{u}_i)^{j-1} \\ u_{i,\bar{u}_k} = 0 \\ F_{i\bar{j},\bar{u}_k} = 0. \end{cases}
\tag{2.6}
$$

The operators $\partial/\partial u$, $\partial/\partial \bar{u}$ also act on the coefficients of the polynomials and define an isomorphism of the ring. The inverse operators are operators of integration. We shall denote by ∂_j^{-1} and $\partial_{\bar{j}}^{-1}$ the integral operators corresponding to $\partial/\partial u_j$ and $\partial/\partial \bar{u}_j$ and which are defined in the obvious fashion.

2.2. The general evolution equation

The class of evolution equations which we consider can be written as

$$u_t = au_{n+1} + b\bar{u}_{n+1} + p^n \tag{2.7}$$

where $p^n \in P[u, \bar{u}]$. The fundamental equation (2.5) becomes,

$$\begin{aligned} &G,_{u_n} = aF_1 + \bar{b}F_{\bar{1}}, \qquad G,_{\bar{u}_n} = \bar{a}F_{\bar{1}} + \mathrm{b}\mathrm{F}_1 \\ &p^nF_1 + \bar{p}^nF_{\bar{1}} - \sum_{i=1}^{n} u_iG,_{u_{i-1}} - \sum_{i=1}^{n} \bar{u}_iG,_{\bar{u}_{i-1}} + [F, G] = 0. \end{aligned} \tag{2.8}$$

From the first relations in (2.8) we obtain

$$G = (aF_1 + \bar{b}F_{\bar{1}})u_n + (\bar{a}F_{\bar{1}} + bF_1)\bar{u}_n + L^{(n-1)}, \qquad L^{(n-1)} \in P[u, \bar{u}, F]. \tag{2.9}$$

The second equation in (2.8) can now be written as

$$\begin{aligned} p^nF_1 + \bar{p}^nF_{\bar{1}} &- \sum_{i=1}^{n} u_iL,_{u_{i-1}}^{(n-1)} - \sum_{i=1}^{n} \bar{u}_iL,_{\bar{u}_{i-1}}^{(n-1)} - u_1((aF_1 + \bar{b}F_{\bar{1}})_1u_n + (\bar{a}F_{\bar{1}} + bF_1)_1\bar{u}_n) \\ &- \bar{u}_1((aF_1 + \bar{b}F_{\bar{1}})_{\bar{1}}u_n + (\bar{a}F_1 + bF_1)_{\bar{1}}\bar{u}_n) \\ &+ [F, (aF_1 + \bar{b}F_{\bar{1}})u_n + (\bar{a}F_{\bar{1}} + bF_1)\bar{u}_n + L^{(n-1)}] = 0. \end{aligned} \tag{2.10}$$

Essentially this consists of a polynomial in u_j, $\bar{u}_j$ which has order n. Thus we can write it as

$$N^{(n,2)} = 0 \qquad \deg_{u_n,\bar{u}_n}(N^{(n,2)}) > 1 \tag{2.11a}$$

$$L,_{u_{n-1}}^{(n-1)} = M^{(n-1,2)}, \qquad L,_{\bar{u}_{n-1}}^{(n-1)} = M^{*(n-1,2)} \tag{2.11b}$$

$$\sum_{i=1}^{n-1} u_iL,_{u_{i-1}}^{(n-1)} + \sum_{i=1}^{n-1} \bar{u}_iL,_{\bar{u}_{i-1}}^{(n-1)} = [F, L^{(n-1)}] + H^{(n-1,1)}. \tag{2.11c}$$

The $N^{(n,2)}$, $M^{(n-1,2)}$, $H^{(n-1,1)}$ are known polynomials and $H^{*(j,k)}_{(u,\bar{u},F)} = \overline{H^{(j,k)}_{(u,\bar{u},F)}}$. Equation (2.11$a$) is either identically satisfied or imposes conditions on the F indeterminates. Since these constitute partial differential equations for F we shall call them F-equations.

If we integrate the equations (2.11b) we obtain

$$L^{(n-1)} = \partial_{u_{n-1}}^{-1}M^{(n-1,2)} + \partial_{\bar{u}_{n-1}}^{-1}M^{*(n-1,2)} + L^{(n-2)}. \tag{2.12}$$

The substitution of this expression into (2.11c) results in a polynomial in u_j, $\bar{u}_j$ of maximal degree $(n-1)$. If we repeat this operation j times (j integrations) then we get

$$\begin{aligned} &N^{(n-j,2+j)} = 0 \qquad \deg_{u_{n-j},\bar{u}_{n-j}} N^{(n-j,2+j)} > 1 \\ &L,_{u_{n-1-j}}^{(n-1-j)} = M^{(n-1-j,2+j)}, \qquad L,_{\bar{u}_{n-1-j}}^{(n-1-j)} = M^{*(n-1-j,2+j)} \\ &\sum_{i=1}^{n-1-j} u_iL,_{u_{i-1}} + \sum_{i=1}^{n-1-j} L,_{\bar{u}_{i-1}}^{(n-1-j)} = [F, L^{(n-1-j)}] + H^{(n-1-j,j+1)}. \end{aligned} \tag{2.13}$$

Thus at each step we produce further F-equations, if $N^{(n-j,2+j)} = 0$ is non-trivial, and a reduction in the order of the quasi-polynomial to be determined. The process terminates when $j = n-1$. The F-equations have to be checked for consistency and F is determined from them up to coefficients $\{X_i\}$ which belong to the prolongation algebra.

Example 1. The simplest non-trivial example which one can consider and which is the subject of detailed analysis in the next section is the equation

$$u_t = au_2 + b\bar{u}_2 + p^{(1)}. \tag{2.14}$$

For this example one finds that

$$G = (aF_1 + \bar{b}F_{\bar{1}})u_1 + (\bar{a}F_{\bar{1}} + bF_1)\bar{u}_1 + L^0. \tag{2.15}$$

For a non-trivial result

$$\begin{aligned} p^{(1)} = {} & \alpha(u, \bar{u})u_1^2 + \beta(u, \bar{u})\bar{u}_1^2 + \gamma(u, \bar{u})u_1\bar{u}_1 \\ & + \delta(u, \bar{u})u_1 + \varepsilon(u, \bar{u})\bar{u}_1 + \lambda(u, \bar{u}). \end{aligned} \tag{2.16}$$

The F-equations are determined from the set

$$\begin{aligned} & \alpha F_1 + \bar{\beta}F_{\bar{1}} - (aF_1 + \bar{b}F_{\bar{1}})_1 = 0 \\ & \beta F_1 + \bar{\alpha}F_{\bar{1}} - (\bar{a}F_{\bar{1}} + bF_1)_{\bar{1}} = 0 \\ & \gamma F_1 + \bar{\gamma}F_{\bar{1}} - (\bar{a}F_{\bar{1}} + bF_1)_1 - (aF_1 + \bar{b}F_{\bar{1}})_{\bar{1}} = 0 \\ & \delta F_1 + \bar{\varepsilon}F_{\bar{1}} + [F, aF_1 + \bar{b}F_{\bar{1}}] = L_1^0 \\ & \varepsilon F_1 + \bar{\delta}F_{\bar{1}} + [F, \bar{a}F_{\bar{1}} + bF_1] = L_{\bar{1}}^0 \\ & \lambda F_1 + \bar{\lambda}F_{\bar{1}} + [F, L^0] = 0. \end{aligned} \tag{2.17}$$

Clearly we cannot hope to solve (2.17) for an arbitrary equation, however for a given equation the equations (2.17) can be solved.

We consider some special subclasses of (2.14) in the following sections.

3. Scaled evolution equations

In this section we look more closely at the class of equations

$$u_t = au_2 + b\bar{u}_2 + K(u, \bar{u}, u_1, \bar{u}_1). \tag{3.1}$$

We discuss various subclasses of (3.1) for which the solution of (2.17) is more transparent. In this way we derive several solvable classes of equation (3.1). We exploit the notion of a complex scaling symmetry: that is, a real scaling symmetry together with a phase symmetry.

Definition. The evolution equation (2.1) is said to have a complex scaling symmetry if it is invariant under the transformation:

$$u \to \lambda^{\alpha} e^{i\theta}u, \qquad x \to \lambda^{\beta}x, \qquad t \to \lambda^{\gamma}t, \qquad \lambda > 0. \tag{3.2}$$

In this section we consider only those equations (3.1) which possess a complex scaling symmetry.

Recall from § 2 that for equation (3.1) to have a non-trivial prolongation structure the function K must be of the form

$$K = k^{(1)}u_1^2 + k^{(2)}\bar{u}_1^2 + k^{(3)}u_1\bar{u}_1 + k^{(4)}u_1 + k^{(5)}\bar{u}_1 + k^{(6)} \tag{3.3}$$

where $k^{(i)} = k^{(i)}(u, \bar{u})$. For (3.2) to be a symmetry of (3.1) with $\gamma = 2\beta$ we need

$a \to a$, $b \to e^{2i\theta} b$

$$\begin{aligned} &k^{(1)} \to e^{-i\theta}\lambda^{-\alpha}k^{(1)} && k^{(2)} \to e^{3i\theta}\lambda^{-\alpha}k^{(2)} \\ &k^{(3)} \to e^{i\theta}\lambda^{-\alpha}k^{(3)} && k^{(4)} \to \lambda^{-\beta}k^{(4)} \\ &k^{(5)} \to e^{2i\theta}\lambda^{-\beta}k^{(5)} && k^{(6)} \to e^{i\theta}\lambda^{\alpha-2\beta}k^{(6)}. \end{aligned} \tag{3.4}$$

We insist that $\theta \neq 0$ but α may be zero. We thus have two immediate cases.

Case 1. When $\alpha = 0$ then $k^{(4)} = k^{(5)} = k^{(6)} = 0$ and equation (3.1) reduces to:

$$u_t = a_1 u_2 + b_1 u^2 \bar{u}_2 + d_1 u^{-1} u_1^2 + d_2 u \bar{u}^{-2} \bar{u}_1^2 + d_3 \bar{u}^{-1} u_1 \bar{u}_1 \tag{3.5}$$

where a_1, b_1 and d_i are functions of $|u|^2$.

Example 2. Wadati *et al* (1979) have shown the equation

$$q_t = i\left(\frac{q}{\sqrt{1 - q\bar{q}}}\right)_{xx} \tag{3.6}$$

to be solvable. When the double derivative is expanded this equation takes the form (3.5). In appendix 3 we rederive the results of Wadati *et al* using the methods of this paper.

Example 3. Nijhoff *et al* (1982) have considered the equation:

$$q_t = i q_2 - (2i/\bar{q}) q_1 \bar{q}_1. \tag{3.7}$$

This is of form (3.5) with $a_1 = i$, $b_1 = d_1 = d_2 = 0$ and $d_3 = -2i$. Substituting these values into (2.17) soon leads to a contradiction. The origin of the contradiction is that condition (2.4) cannot be satisfied. For equation (3.7) we need $F(u, \bar{u}, u_1, \bar{u}_1)$. The corresponding calculation will be found in appendix 2. The linear problem is found to be

$$\begin{aligned} &\phi_x = F\phi, \qquad \phi_t = G\phi \\ &F = \frac{q_1}{\bar{q}} e_+ - \frac{\bar{q}_1}{q} e_- + \lambda h \\ &G = i\left(\frac{q_2}{\bar{q}} - \frac{2}{q\bar{q}} q_1 \bar{q}_1 + \frac{1}{\bar{q}^2} q_1 \bar{q}_1 + 2\lambda \frac{q_1}{\bar{q}}\right) e_+ \\ &\qquad + i\left(\frac{\bar{q}_2}{q} - \frac{2}{q\bar{q}} q_1 \bar{q}_1 + \frac{1}{q^2} q_1 \bar{q}_1 - 2\lambda \frac{\bar{q}_1}{q}\right) e_- + i\left(\lambda^2 + \frac{2}{q\bar{q}} q_1 \bar{q}_1\right) h \end{aligned} \tag{3.8}$$

where $e_\pm$, h are the basis (1.9) for sl(2, C).

It is evident that we can make the substitution $Q = q_1/\bar{q}$. Q is found to satisfy the NLS equation

$$Q_t = i Q_2 - 2i Q^2 \bar{Q}. \tag{3.9}$$

Finally, we consider the restricted class of case 1 for which $b_1 = 0$. The transformation $u = e^v$ gives an equation which does not admit a phase symmetry:

$$v_t = a v_2 + (a + f_1) v_1^2 + f_2 \bar{v}_1^2 + f_3 v_1 \bar{v}_1 \tag{3.10}$$

where the coefficients a, f_i are functions of the sum $v+\bar{v}$. There exists a viable real restriction: $f_2=f_3=0$, a and f_1 real constants gives the potential Burger's equation.

Case 2. When $\alpha \neq 0$ then a is constant, $b=0$ and each $k^{(i)}$ is determined up to a single constant:

$$k^{(1)}=d_1u^{-1}, \qquad k^{(2)}=d_2u\bar{u}^{-2}, \qquad k^{(3)}=d_3\bar{u}^{-1} \tag{3.11a}$$

$$\begin{aligned} k^{(4)}&=\begin{cases} c_1u^n\bar{u}^n & \text{if } \beta=-2n\alpha \\ 0 & \text{otherwise} \end{cases} \\ k^{(5)}&=\begin{cases} c_2u^{m+2}\bar{u}^m & \text{if } \beta=-2(m+1)\alpha \\ 0 & \text{otherwise} \end{cases} \\ k^{(6)}&=\begin{cases} c_3u^{l+1} & \text{if } \beta=-l\alpha \\ 0 & \text{otherwise} \end{cases} \\ & \qquad c_i, d_i \in C, \qquad \alpha \neq 0. \end{aligned} \tag{3.11b}$$

The equations (3.11b) are mutually consistent if we take $l=2n$, $m=n-1$.

Thus the most general scale invariant form of (3.1), with $\alpha \neq 0$, is given by

$$\begin{aligned} u_t&=au_2+d_1u^{-1}u_1^2+d_2u\bar{u}^{-2}\bar{u}_1^2+d_3\bar{u}^{-1}u_1\bar{u}_1 \\ &\quad +c_1u^n\bar{u}^nu_1+c_2u^{n+1}\bar{u}^{n-1}\bar{u}_1+c_3u^{2n+1}\bar{u}^{2n}. \end{aligned} \tag{3.12}$$

It is possible to substitute the above forms for $k^{(i)}$ in the prolongation equations (2.17). The first three equations of (2.17) are homogeneous in u and $\bar{u}$, so can be solved by taking linear combinations of solutions $u^\alpha\bar{u}^\beta$ for allowable α and β. However, this is rather tedious and not very illuminating. We present an interesting subclass for which $d_i\equiv 0$.

When $d_i\equiv 0$, F is easily found to be:

$$F=X_1u+X_2\bar{u}+X_3u\bar{u}+X_4 \tag{3.13}$$

and if $a+\bar{a}\neq 0$ then $X_3=0$. In addition we find that

$$\begin{aligned} L^0&=\frac{c_2}{n}u^{n-1}\bar{u}^nX_1+\frac{\bar{c}_1}{n+1}\bar{u}^{n+1}u^nX_2+\frac{\bar{u}^{n+1}u^{n+1}}{n+1}(c_2+\bar{c}_1)X_3 \\ &\quad +\bar{a}u\bar{u}[X_1, X_2]-au[X_1, X_4]-\bar{a}u\bar{u}[X_3, X_4] \\ &\quad +\bar{a}u^2\bar{u}[X_1X_3]+X_0 \end{aligned} \tag{3.14a}$$

$$\begin{aligned} &c_3u^{2n+1}\bar{u}^{2n}X_1+\bar{c}_3\bar{u}^{2n+1}u^{2n}X_2+(c_3+\bar{c}_3)u^{2n+1}\bar{u}^{2n+1}X_3 \\ &\quad +[X_1u+X_2\bar{u}+X_3u\bar{u}+X_4, L^0]=0. \end{aligned} \tag{3.14b}$$

The symmetry (3.2) imposes transformation properties on the generators $\{X_i\}_{i=0}^4$. With the conditions (3.11) these are:

$$\begin{aligned} &X_0\to\lambda^{4n\alpha}X_0, \qquad X_1\to\lambda^{(2n-1)\alpha}\,\mathrm{e}^{-\mathrm{i}\theta}X_1, \qquad X_2\to\lambda^{(2n-1)\alpha}\,\mathrm{e}^{\mathrm{i}\theta}X_2. \\ &X_3\to\lambda^{2(n-1)\alpha}X_3, \qquad X_4\to\lambda^{2n\alpha}X_4. \end{aligned} \tag{3.15}$$

The integrability conditions for L^0 give the following three classes of constraints on the prolongation algebra.

(I) $n=0$

$$\begin{gathered} c_2X_1=0 \qquad \bar{c}_2X_2=0 \\ ((c_1+\bar{c}_2)-(c_2+\bar{c}_1))X_3-(a+\bar{a})[X_1,X_2]-(a-\bar{a})[X_3,X_4]=0 \\ [X_1,X_3]=0 \qquad [X_2,X_3]=0 \end{gathered} \tag{3.16}$$

(II) $n=1$

$$\begin{gathered} (c_1-2c_2)X_1-2\bar{a}[X_1,X_3]=0 \\ (2\bar{c}_2-\bar{c}_1)X_2+2a[X_2,X_3]=0 \\ ((c_1+\bar{c}_2)-(c_2+\bar{c}_1))X_3=0 \\ (a+\bar{a})[X_1,X_2]+(a-\bar{a})[X_3,X_4]=0 \end{gathered} \tag{3.17}$$

(III) $n\neq 0, 1$

$$\begin{gathered} (nc_1-(n+1)c_2)X_1=0, \qquad ((n+1)\bar{c}_2-n\bar{c}_1)X_2=0 \\ ((c_1+\bar{c}_2)-(c_2+\bar{c}_1))X_3=0 \\ (a+\bar{a})[X_1,X_2]+(a-\bar{a})[X_3,X_4]=0 \\ [X_2,X_3]=0 \qquad [X_1,X_3]=0. \end{gathered} \tag{3.18}$$

It is necessary to consider each of the cases (I)–(III) in turn.

Case I. Here the equation takes the form

$$u_t=au_2+c_1u_1+c_2u\bar{u}^{-1}\bar{u}_1+c_3u. \tag{3.19}$$

By using $u\to e^{-c_3t}u$ we can transform c_3 to zero. By a real Galilean transformation $x\to x+\beta t$, $\beta=\mathrm{Re}(c_1)$ we can make c_1 pure imaginary. For a nonlinear equation we require $c_2\neq 0$. Thus (3.16) implies $X_1=X_2=0$. Recall that $a+\bar{a}\neq 0\Rightarrow X_3=0$, which would lead to a trivial prolongation structure. Thus $a+\bar{a}=0$. The remaining elements $\{X_0, X_3, X_4\}$ form a complete Lie algebra:

$$\begin{gathered} [X_0,X_3]=-(c_2+\bar{c}_1+a\alpha)X_3 \\ [X_4,X_3]=-X_3, \qquad [X_0,X_4]=0 \end{gathered} \tag{3.20}$$

where $\alpha=(1/2a)(c_2+\bar{c}_1-c_1-\bar{c}_2)$. The single Jacobi identity is satisfied and leads to no restrictions on the parameters. These elements just form the lower triangular, solvable subalgebra of sl$(2, C)$:

$$X_3=e_-, \qquad X_4=\tfrac{1}{2}\alpha h+\lambda^{2\alpha}e_-, \qquad X_0=(c_2+\bar{c}_1+a\alpha)X_4 \tag{3.21}$$

where e_- and h are given by (1.7). The corresponding linear problem

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=\begin{pmatrix}\frac{1}{2}\alpha & 0\\ \lambda^{2\alpha}+u\bar{u} & -\frac{1}{2}\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{3.22}$$

cannot be used to construct the whole function u but only the square of its modulus $u\bar{u}$. Furthermore, with $a+\bar{a}=0$ it is not possible to combine (3.19) and its complex conjugate to derive an equation autonomous in $u\bar{u}$.

Case II. Here the equation takes the form

$$u_t = au_2 + c_1 u\bar{u}u_1 + c_2 u^2 \bar{u}_1 + c_3 u^3 \bar{u}^2. \tag{3.23}$$

The integrability conditions (3.14) lead to (3.17) and

$$\begin{aligned}
&c_3 X_1 - \tfrac{1}{2}(c_2 - \bar{c}_1)[X_1, X_3] + \bar{a}[X_3[X_1, X_3]] = 0 \\
&\bar{c}_3 X_2 + \tfrac{1}{2}c_2[X_2, X_3] = 0 \qquad (c_3 + \bar{c}_3)X_3 = 0 \\
&(c_2 - \tfrac{1}{2}\bar{c}_1)[X_1, X_2] + \tfrac{1}{2}(\bar{c}_1 + c_2)[X_3, X_4] - \bar{a}[X_2, [X_1, X_3]] \\
&\qquad - a[X_3, [X_1, X_2]] + \bar{a}[X_3[X_3, X_4]] = 0 \\
&\bar{a}[X_1, [X_1, X_2]] - \bar{a}[X_1[X_3, X_4]] + \bar{a}[X_4, [X_1, X_3]] \\
&\qquad - a[X_3, [X_1, X_4]] - c_2[X_1, X_4] = 0 \\
&\bar{a}[X_2, [X_1, X_2]] - \bar{a}[X_2, [X_3, X_4]] - \bar{a}[X_3, [X_2, X_4]] + \tfrac{1}{2}\bar{c}_1[X_4, X_2] = 0 \\
&[X_1, [X_1, X_3]] = [X_1, [X_1, X_4]] = [X_2, [X_2, X_4]] = 0
\end{aligned} \tag{3.24a}$$

$$\begin{aligned}
&[X_0, X_1] = a[X_4, [X_4, X_1]] \qquad [X_0, X_2] = \bar{a}[X_4, [X_4, X_2]] \\
&[X_0, X_3] = \bar{a}[X_4, [X_1, X_2]] + \bar{a}[X_4, [X_4, X_3]] \\
&\qquad - \bar{a}[X_1, [X_2, X_4]] - a[X_2, [X_1, X_4]] \\
&[X_0, X_4] = 0.
\end{aligned} \tag{3.24b}$$

There are two subcases to consider: $c_3 \neq 0$ and $c_3 = 0$.

Case IIa. $c_3 \neq 0$. First we dismiss the case $a + \bar{a} \neq 0$, for then we have $X_3 = 0$, which immediately implies $X_1 = X_2 = 0$ and a trivial prolongation structure. We thus have

$$\begin{aligned}
&a + \bar{a} = 0, \qquad X_3 \neq 0, \qquad [X_3, X_4] = 0 \\
&c_1 + \bar{c}_2 = c_2 + \bar{c}_1, \qquad c_3 + \bar{c}_3 = 0 \\
&[X_1, X_3] = -(1/2a)(c_1 - 2c_2)X_1, \qquad [X_2, X_3] = (1/2a)(\bar{c}_1 - 2\bar{c}_2)X_2
\end{aligned} \tag{3.25}$$

$$c_2[X_2, X_3] = -2\bar{c}_3 X_2. \tag{3.25}$$

We assume $c_2 \neq 0$ (the case $c_2 = 0$ is trivial), so that $4a\bar{c}_3 = c_2(2\bar{c}_2 - \bar{c}_1)$, which, by the reality of $a\bar{c}_3$, leads to

$$(c_1 - c_2)(c_2 - \bar{c}_2) = 0. \tag{3.26}$$

The elements $\{X_i\}_1^4$ must satisfy (3.25) and

$$\begin{aligned}
&[X_3, [X_1, X_2]] = (1/2a)(\bar{c}_1 - c_1)[X_1, X_2], [X_3, [X_1, X_4]] \\
&[X_3, [X_1, X_4]] = (1/2a)(c_1 - 2c_2)[X_1, X_4] \\
&[X_1, [X_1, X_2]] = (1/a)(c_2 - c_1)[X_1, X_4], [X_2, [X_2, X_1]] \\
&[X_2, [X_2, X_1]] = (1/a)(\bar{c}_1 - \bar{c}_2)[X_2, X_4] \\
&[X_1, [X_1, X_4]] = [X_2, [X_2, X_4]] = 0.
\end{aligned} \tag{3.27}$$

The remaining element X_0 is defined by (3.24*b*). The resulting algebra can be represented in terms of the basis (1.9) of sl(2, C):

$$X_1 = \rho\lambda e_-, \qquad X_2 = \sigma\lambda e_+, \qquad X_3 = (1/4a)(2c_2 - c_1)h$$
$$X_4 = \tau\lambda^2 h, \qquad X_0 = -2a\tau^2\lambda^4 h \tag{3.28}$$

where $\rho\sigma = (1/a)(c_1 - c_2)\tau$. We have taken into account the transformation properties (3.15) (with $\alpha = 1$) of the prolongation algebra and the automorphism (1.11*a*) and (1.11*b*) of sl(2, C). This sl(2, C) closure forces both parameters c_1 and c_2 to be real. In summary, the equation ($a = \mathrm{i}$):

$$u_t = \mathrm{i}u_2 + c_1 u\bar{u}u_1 + c_2 u^2\bar{u}_1 - \tfrac{1}{4}\mathrm{i}c_2(2c_2 - c_1)u^3\bar{u}^2 \tag{3.29}$$

is isospectral to the linear scattering problem:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}\tau\lambda^2 - \frac{1}{4}\mathrm{i}(2c_2 - c_1)u\bar{u} & \sigma\lambda\bar{u}\\ \rho\lambda u & -\tau\lambda^2 + \frac{1}{4}\mathrm{i}(2c_2 - c_1)u\bar{u}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{3.30}$$

where the parameters ρ, σ and τ can be chosen at will, subject to the condition following (3.28). Notice that the case $c_1 = c_2$ is degenerate and corresponds to $\sigma = 0$ and therefore $X_2 = 0$. Equation (3.29) represents a 2-parameter family of isospectral flows of (3.30). The special case of $c_1 = 2c_2$ is the DNLS equation discussed by Kaup and Newell (1978).

Case IIb. $c_3 = 0$. This condition immediately implies that: $c_2[X_2, X_3] = 0$. The case $c_2 = 0$ is the DNLS equation (1.4) of Chen *et al* (1979). This was considered in our previous paper I. With $a = \mathrm{i}$ and $c_1 = -2$, the corresponding linear problem is

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}\frac{1}{2}\mathrm{i}(\lambda^2 + u\bar{u}) & \lambda u\\ \lambda\bar{u} & -\frac{1}{2}\mathrm{i}(\lambda^2 + u\bar{u})\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{3.31}$$

We thus consider the case $c_2 \neq 0$. This immediately implies

$$X_3 = 0, \qquad c_1 = 2c_2. \tag{3.32}$$

In order to obtain a non-trivial algebra we must have $a + \bar{a} = \bar{c}_2 - c_2 = 0$. With condition (3.32) equation (3.23) reduces to the DNLS equation of Kaup and Newell (1978), so we omit the details.

Case III. This is the case of $n \neq 0, 1$. When $n \neq \frac{1}{2}$ the prolongation algebra proves to be trivial. Therefore we have $n = \frac{1}{2}$. For a non-trivial algebra we need $c_1 = 3c_2$, $X_3 = 0$. Equation (3.12) takes the form:

$$u_t = au_2 + 2c_2(u^{3/2}\bar{u}^{1/2})_1 + c_3 u^2\bar{u}. \tag{3.33}$$

If $c_2 = 0$ then (3.33) reduces to the well known NLS equation (1.1). Therefore we assume $c_2 \neq 0$. In this case for a non-trivial algebra we require $X_0 = X_4 = 0$, $a + \bar{a} = 0$. The algebra then reduces to:

$$a[X_1, [X_1, X_2]] = c_3 X_1, \qquad a[X_2, [X_1, X_2]] = \bar{c}_3 X_2$$
$$(3c_2 - \bar{c}_1)[X_1, X_2] = 0. \tag{3.34}$$

An sl(2, $\mathscr{C}$) representation of the algebra is given by

$$X_1 = \lambda^{-1}\rho e_-, \qquad X_2 = \lambda\sigma e_+, \qquad \rho\sigma = -c_3/2a,$$
$$\bar{c}_3 + c_3 = 0, \qquad \bar{c}_2 - c_2 = 0. \tag{3.35}$$

It is interesting to observe that in this case the spectral parameter arises from the phase symmetry $u \to \lambda u$, $\bar{u} \to \lambda^{-1}\bar{u}$. The equation (3.33), $c_2 \neq 0$, is thus isospectral to the linear scattering problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} 0 & \sigma\lambda\bar{u} \\ \rho\lambda^{-1}u & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.36}$$

4. Conclusions

In this paper we have extended the ideas of I from real to complex evolution equations. We presented an algorithm and applied it to second-order complex equations of NLS type. However, for the general equation (2.14) we only 'set up' the problem, presenting the system of equations (2.17) to be satisfied by the various functions occurring in (2.15) and (2.16). We did not solve these equations in such a general context. In § 3 we considered scaled, quasi-polynomial flows and calculated some interesting new examples. However, even here we did not exhaust all the possibilities. In the case of (3.5) we only gave a few specific examples. In the case of (3.12) we exhausted the possibilities when $d_i = 0$ but did not touch upon the general situation. Thus, there is still much to do even in the case of scaled equations.

However, there are generalised scaled equations not included in § 3 at all. For instance, integrable deformations of the scaled equations of § 3 would contain scaled parameters. For instance, the deformed NLS equation which is the second-order flow of

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x \begin{pmatrix} \lambda(1-\varepsilon^2 u\bar{u})^{1/2} & (1-\lambda^2\varepsilon^2)^{1/2}u \\ -(1-\lambda^2\varepsilon^2)^{1/2}\bar{u} & -\lambda(1-\varepsilon^2 u\bar{u})^{1/2} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.1}$$

has the correct scaling properties provided $\varepsilon \to \lambda^{-1}\varepsilon$. This is the complex version of (4.4.12) in I. We consider the prolongation structure of deformed equations in another paper (Dodd and Fordy 1984).

Finally we mention that if a complex equation admits a phase symmetry, even a discrete phase symmetry, (see the example of appendix 4) then it seems likely that the representation of the simple Lie algebra can be immediately determined. In the scaled equations of § 3 the simple algebra is sl(2, C) whereas for the example of appendix 4 it is sl(3, C). We intend to investigate this further elsewhere.

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Appendix 1. Generalised symmetries

In this appendix we consider those functions K for which (3.1) (with $b \equiv 0$, constant a) admits a generalised symmetry. To do this we try to construct an evolution equation which commutes with (3.1). Such evolution equations are expected to exist from the general theory of commuting isospectral flows of a given eigenvalue problem.

We start with the second-order equation

$$u_t = au_2 + K(u, \bar{u}, u_1, \bar{u}_1) \tag{A1.1}$$

and its complex conjugate. We seek a commuting third-order flow:

$$u_\tau = f(u, \bar{u}, \ldots, u_3, \bar{u}_3) \tag{A1.2}$$

and its complex conjugate. For (A1.1) and (A1.2) to commute $(u_{t\tau} = u_{\tau t})$, f satisfies:

$$D_t f = aD^2 f + \sum_{i=0}^{1} \left(D^i f \frac{\partial}{\partial u_i} + D^i \bar{f} \frac{\partial}{\partial \bar{u}_i} \right) K \tag{A1.3}$$

where

$$D \equiv D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}$$

and

$$D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} u_{it} \frac{\partial}{\partial u_i}$$

are the total x and t derivatives respectively.

Remark. Although it is to be expected that (A1.1) will possess an infinite number of commuting flows (if it is solvable) there is no guarantee that it has a third-order commuting flow. We have thus restricted attention to a subclass of equations (A1.1). We further assume that f be of the form

$$f = cu_3 + H(u, \bar{u}, \ldots, u_2, \bar{u}_2). \tag{A1.4}$$

The functions H and K must satisfy

$$\begin{aligned} aD^2 H &+ \sum_{i=0}^{1} (cu_{3+i} + D^i H) K_{,u_i} + \sum_{i=0}^{1} (\bar{c}\bar{u}_{3+i} + D^i \bar{H}) K_{,\bar{u}_i} \\ &= cD^3 K + \sum_{i=0}^{2} (cu_{3+i} + D^i K) H_{,u_i} + \sum_{i=0}^{2} (\bar{a}\bar{u}_{2+i} + D^i \bar{K}) H_{,\bar{u}_i}. \end{aligned} \tag{A1.5}$$

This system of equations is rather like (2.5) and is solved by a similar recursive procedure. Since H and K contain nothing higher than u_2 and $\bar{u}_2$, we start by equating coefficients of u_3 and $\bar{u}_3$, giving:

$$H = \frac{3c}{2a} K_{,u_1} u_2 + \frac{3c}{4a^2}(a + \bar{a}) K_{,\bar{u}_1} \bar{u}_2 + h(u, \bar{u}, u_1, \bar{u}_1). \tag{A1.6}$$

The system (A1.5) is highly overdetermined, so with enough stamina we could carry the calculation through to the end. We would find an H given purely in terms of K together with some differential conditions on K, telling us which equations are (most probably) solvable. This type of calculation has been performed by Ibragimov and Shabat (1980) and Fokas (1980) for second- and third-order equations for one real function u. The number of possible cases turns out to be enormous.

For systems of equations, such as our complex equation, this method is far less practical. Furthermore, to obtain the linear problem we would still have to do the prolongation calculation. It is thus simpler to go straight to the prolongation calculation. Once the linear problem has been found the whole hierarchy of commuting flows can be generated in the usual way.

Appendix 2

In this appendix we consider the prolongation structure for the equation (3.7) (Nijhoff *et al* 1982)

$$q_t = \mathrm{i}q_2 + \mathrm{i}f(q, \bar{q})q_1\bar{q}_1$$

where

$$f(q, \bar{q}) = -2/\bar{q}. \tag{A2.1}$$

The equation admits the phase symmetry

$$q \to \mathrm{e}^{\mathrm{i}\theta}q, \qquad \bar{q} \to \mathrm{e}^{-\mathrm{i}\theta}q. \tag{A2.2}$$

If we assume, as we have done throughout the paper, that $F = F(q, \bar{q})$ then we rapidly find that

$$F = q\bar{q}X_1 + qX_2 + \bar{q}X_3 + X_4 \tag{A2.3}$$

and that the constraints on the algebra require that $X_2 = X_3 = 0$. This results, after considering the scaling, in a trivial prolongation structure. This equation furnishes an example where it is necessary to include the first-order derivatives in F in order to obtain a non-trivial WE-prolongation,

$$F = F(q, q_1, \bar{q}, \bar{q}_1). \tag{A2.4}$$

Cases such as these have been excluded from I and the present paper. In this case we obtain from the equation

$$D_tF - D_xG + [F, G] = 0 \tag{A2.5}$$

the following information

$$\begin{aligned}
F &= A(q, \bar{q})q_1\bar{q}_1 + B(q, \bar{q})q_1 + C(q, \bar{q})\bar{q}_1 + D(q, \bar{q}) \\
G &= \mathrm{i}(A(q, \bar{q})\bar{q}_1 + B(q, \bar{q}))q_2 - \mathrm{i}(A(q, \bar{q})q_1 \\
&\quad + C(q, \bar{q}))\bar{q}_2 + g(q_1, \bar{q}_1, q, \bar{q}) \\
g_{,q_1} &= -\mathrm{i}A\bar{f}q_1\bar{q}_1 + \mathrm{i}\bar{q}_1(Bf - C\bar{f} + [D, A] + [C, B] + C_1 - B_{\bar{1}}) \\
g_{,\bar{q}_1} &= \mathrm{i}(fA - 2[A, C])q_1\bar{q}_1 + \mathrm{i}(A_1 - A\bar{f} - [B, A])q_1^2 \\
&\quad + \mathrm{i}(C_1 - B_{\bar{1}} + Bf - C\bar{f} - [D, A] - [B, C])q_1 + \mathrm{i}(-D_{\bar{1}} + [C, D]) \\
&\mathrm{i}F_1fq_1\bar{q}_1 - \mathrm{i}F_{\bar{1}}\bar{f}q_1\bar{q}_1 + \mathrm{i}F_{,q_1}(f_1q_1^2\bar{q}_1 + f_{\bar{1}}q_1\bar{q}_1^2) \\
&\quad - \mathrm{i}F_{,\bar{q}_1}((\bar{f}_1)\bar{q}_1^2q_1 + (\bar{f}_{\bar{1}})\bar{q}_1q_1^2) + [F, g] - g_{\bar{1}}\bar{q}_1 - g_1q_1 = 0.
\end{aligned} \tag{A2.6}$$

The integrability conditions on g give

$$\begin{aligned}
&2A_1 - A\bar{f} - 2[B, A] = 0 \\
&[D, A] = 0 \\
&2A_{\bar{1}} - Af = 0.
\end{aligned} \tag{A2.7}$$

The last equation of (A2.7) integrates to give (from scaling)

$$A = -X_1/|q|^2 \tag{A2.8}$$

and

$$D = dX_1, \qquad B = \bar{q}bX_1 \tag{A2.9}$$

where $d = d(qq)$, $b = b(q\bar{q})$ provided $A \neq 0$. Thus we see from (A2.7) that $A = 0$ and

$$\begin{aligned} g = \mathrm{i}\bar{q}_1 q_1(Bf - C\bar{f} + [C, B] + C_1 - B_{\bar{1}}) \\ + \mathrm{i}q_1(D_1 + [D, B]) - \mathrm{i}\bar{q}_1(D_{\bar{1}} + [D, C]) + \mathrm{i}E \end{aligned} \tag{A2.10}$$

where $E = E(q, \bar{q})$. From (A2.10) and (A2.6) we obtain a system of PDE's in the unknowns B, C, D and E. Moreover they involve the Lie bracket of the algebra and so are not simple to solve. We adopt the strategy of I in order to solve the PDE's. We assume a homomorphism of the prolongation algebra into a simple Lie algebra starting with sl(2, $\mathbb{C}$). If the underlying simple algebra is sl(2, $\mathbb{C}$) then the scaling suggests

$$F = bq_1 e_+ + c\bar{q}_1 e_- + dh. \tag{A2.11}$$

We then obtain from (A2.6) a series of PDE's for b, c, d and e. These equations are easily integrated and using the scaling symmetry we find that

$$b = \bar{q}^{-1}, \qquad c = -\bar{q}^1, \qquad d = \lambda \qquad \text{and} \qquad E = \lambda^2 h \tag{A2.12}$$

Piecing the information together, we finally obtain the linear problem (3.8).

Gauge Transformation. Suppose we have a single real equation, isospectral to

$$\phi_x = F\phi, \qquad F = A(q) + B(q)q_x. \tag{A2.13}$$

If $\hat{\phi} = T(q)\phi$ is a gauge transformation we have

$$\hat{F}(q, q_x)T = T(A + Bq_x) + T_q q_x \tag{A2.14}$$

since $T_x = T_q T_x$. Thus, with $F(q, q_x)$ linear in q_x we can choose $T(q)$ so that

$$T_q + TB = 0 \tag{A2.15}$$

and make $\hat{F}$ independent of q_x. Thus whenever F depends upon q_x *linearly* we can gauge this dependence away. Similarly, if F depends upon many derivatives then the dependence upon the highest can be gauged away provided this derivative occurs only linearly.

Unfortunately, the above manipulation may not be possible when dealing with a system of real equations or even one complex equation, for then:

$$\phi_x = F\phi, \qquad F = A(q, \bar{q}) + B(q, \bar{q})q_x + C(q, \bar{q})\bar{q}_x. \tag{A2.16}$$

To remove q_x and $\bar{q}_x$ by gauge transformation $\hat{\phi} = T(q, \bar{q})\phi$ we need

$$T_q + TB = 0, \qquad T_q + TC = 0 \tag{A2.17}$$

and this is only possible if the following integrability conditions are satisfied

$$B_{\bar{q}} - C_q + [B, C] = 0. \tag{A2.18}$$

Since (A2.11) does not satisfy these conditions, we cannot gauge away the q_1 and $\bar{q}_1$ dependence of F.

Appendix 3

We derive, using the prolongation method, the isospectral problem for the equation (3.6) Wadati *et al* 1979)

$$q_t = \mathrm{i} D_x^2\left(\frac{q}{\sqrt{1-|q|^2}}\right). \tag{A3.1}$$

Put $f(q\bar{q}) = (1-|q|^2)^{-1/2}$ then equation (A3.1) can be written as

$$q_t = \mathrm{i}(aq_2 + b\bar{q}_2 + cq_1^2 + dq_1\bar{q}_1 + e\bar{q}_1^2)$$

where

$$a = \tfrac{1}{2}f(2+q\bar{q}f^2) \qquad b = \tfrac{1}{2}f^3q^2 \qquad c = \bar{q}f^3(1+\tfrac{3}{4}q\bar{q}f^2)$$
$$d = qf^3(2+\tfrac{3}{2}q\bar{q}f^2) \qquad e = \tfrac{3}{4}q^3f^5. \tag{A3.2}$$

We find that

$$G = \mathrm{i}(aF_1 - \bar{b}F_{\bar{1}})q_1 + \mathrm{i}(bF_1 - \bar{a}F_{\bar{1}})\bar{q}_1 + g(q, \bar{q})$$

and that three of the F-equations are

$$(\tfrac{1}{2}(1-q\bar{q})q\bar{q}-1)F_2 - \tfrac{1}{2}(1-q\bar{q})\bar{q}^2F_{1\bar{1}} = 0$$
$$(\tfrac{1}{2}(1-q\bar{q})q\bar{q}-1)F_{\bar{2}} - \tfrac{1}{2}(1-q\bar{q})q^2F_{1\bar{1}} = 0 \tag{A3.3}$$
$$dF_1 - \bar{d}F_{\bar{1}} - (bF_1 - \bar{a}F_{\bar{1}})_1 - (aF_1 - \bar{b}F_{\bar{1}})_{\bar{1}} = 0.$$

The solution to the F equations (A3.3) is given by

$$F = qX_1 + \bar{q}X_2 + X_3. \tag{A3.4}$$

The remaining F-equations define the prolongation algebra

$$[X_1q + X_2\bar{q} + X_3, -2f[X_1, X_2] + \bar{q}f[X_2, X_3] - qf[X_1, X_3] + X_0] = 0. \tag{A3.5}$$

Scaling arguments then give

$$F = \mathrm{i}\lambda h + \lambda q e_+ + \lambda\bar{q}e_-$$
$$G = \mathrm{i}(1-q\bar{q})^{-3/2}\{-2\lambda^2(1-q\bar{q})h + \tfrac{1}{2}\lambda((2-q\bar{q})q_1 + q^2\bar{q}_1$$
$$-4\mathrm{i}\lambda q(1-q\bar{q}))e_+ - \tfrac{1}{2}\lambda((2-q\bar{q})\bar{q}_1 + \bar{q}^2q_1 - 4\mathrm{i}\lambda\bar{q}(1-q\bar{q}))e_-\} \tag{A3.6}$$

which is the linear problem given in Wadati *et al* (1979).

Appendix 4

In this appendix we consider the prolongation structure for the equation

$$q_t = \mathrm{i}q_2 + D_x(\bar{q}^2) \tag{A4.1}$$

which was introduced by Mikhailov and Zakharov (see Mikhailov 1981). This presents an example where the representation is obtained from a homomorphism of the prolongation algebra into sl(3, C).

From the example considered in § 2, equations (2.14)–(2.17), we quickly find that,

$$\begin{aligned} &F = X_1 q + X_2 \bar{q} + X_3 q\bar{q} + X_4 \\ &L_1^0 = 2qF_{\bar{1}} + \mathrm{i}[F, F_1] \\ &L_{\bar{1}}^0 = 2\bar{q}F_1 - \mathrm{i}[F, F_{\bar{1}}] \\ &[F, L^0] = 0. \end{aligned} \tag{A4.2}$$

The two equations for L^0 can be integrated to give

$$L^0 = \bar{q}^2 X_1 + q^2 X_2 - \mathrm{i}q[X_1, X_4] - \mathrm{i}q\bar{q}[X_1, X_2] + \mathrm{i}\bar{q}[X_2, X_4] + X_0. \tag{A4.3}$$

It is also found that the integrability conditions require that $X_3 \equiv 0$. Consequently we have

$$G = (\mathrm{i}q_1 + \bar{q}^2)X_1 + (-\mathrm{i}\bar{q}_1 + q^2)X_2 - \mathrm{i}q[X_1, X_4] - \mathrm{i}q\bar{q}[X_1, X_2] + \mathrm{i}\bar{q}[X_2, X_4] + X_0 \tag{A4.4}$$

and the prolongation algebra is defined by

$$\begin{aligned} &[X_2, X_4] + \mathrm{i}[X_1, [X_1, X_4]] = 0 \\ &[X_1, X_4] - \mathrm{i}[X_2, [X_2, X_4]] = 0 \\ &[X_1, X_0] - \mathrm{i}[X_4, [X_1, X_4]] = 0 \\ &[X_2, X_0] + \mathrm{i}[X_4, [X_2, X_4]] = 0 \\ &[X_1, X_2] = 0, [X_4, X_0] = 0. \end{aligned} \tag{A4.5}$$

The equation (A4.1) admits the scaling symmetry

$$x \to \lambda^{-1} x, \qquad t \to \lambda^{-2} t, \qquad q \to \lambda q \tag{A4.6}$$

so that the algebra has the following automorphism

$$X_1 \to X_1, \qquad X_2 \to X_2, \qquad X_4 \to \lambda X_4, \qquad X_0 \to \lambda^2 X_0. \tag{A4.7}$$

If we attempt to find a homomorphism of (A4.5) into sl(2, C) then we quickly find that there is only the trivial solution (X_1 and X_2 have to be proportional to h). In fact a homomorphism of the prolongation algebra into sl(3, C) is easy to find because (A4.1) admits a discrete phase symmetry,

$$q \to \mu^2 q \quad \text{where } \mu^3 = 1. \tag{A4.8}$$

Then we find that

$$X_1 \to \mu X_1, \qquad X_2 \to \mu^{-1} X_2, \qquad X_4 \to X_4, \qquad X_0 \to X_0. \tag{A4.9}$$

Then if we consider the transformation of the simple roots of sl(3, C) according to

$$e_{\alpha_i} \to \mu e_{\alpha_i} \qquad e_{-\alpha_i} \to \mu^{-1} e_{-\alpha_i} \tag{A4.10}$$

sl(3, C) has the natural scaling

$$\begin{pmatrix} \mu^0 & \mu^1 & \mu^{-1} \\ \mu^{-1} & \mu^0 & \mu^1 \\ \mu^1 & \cdot\mu^{-1} & \mu^0 \end{pmatrix}. \tag{A4.11}$$

It is then easy to check that

$$X_1=\frac{1}{\sqrt{3}}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\qquad X_2=\frac{1}{\sqrt{3}}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\tag{A4.12}$$

and then from a particular choice of the scaling (A4.7) and the usual cyclic requirement,

$$X_0=cX_4^2,\qquad X_4^3=I\tag{A4.13}$$

so that a calculation quickly gives

$$X_4=\begin{pmatrix}\mu&0&0\\0&\mu^{-1}&0\\0&0&1\end{pmatrix},\qquad X_0=-\sqrt{3}\begin{pmatrix}\mu^{-1}&0&0\\0&\mu&0\\0&0&1\end{pmatrix}.\tag{A4.14}$$

The parameter can be put into the problem using (A4.7)

$$\Phi_x=\left[\frac{1}{\sqrt{3}}\begin{pmatrix}0&q&\bar{q}\\\bar{q}&0&q\\q&\bar{q}&0\end{pmatrix}+\lambda\begin{pmatrix}\mu&0&0\\0&\mu&0\\0&0&1\end{pmatrix}\right]\Phi.\tag{A4.15}$$

Remark. Equation (A4.1) is a transformed version of the modified Boussinesz equation (Fordy and Gibbons 1981). The linear problem (A4.15) is just that of the third-order modified Lax system but written in the circulant basis (Kupershmidt and Wilson 1981).

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